

# Position-dependent noncommutativity in quantum mechanics

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## Abstract

The model of the position-dependent noncommutativity in quantum mechanics is proposed. We start with a given commutation relations between the operators of coordinates  $[\hat{x}^i, \hat{x}^j] = \omega^{ij}(\hat{x})$ , and construct the complete algebra of commutation relations, including the operators of momenta. The constructed algebra is a deformation of a standard Heisenberg algebra and obey the Jacobi identity. The key point of our construction is a proposed first-order Lagrangian, which after quantization reproduces the desired commutation relations. Also we study the possibility to localize the noncommutativity.

## 1 Introduction

Recently quantum field theory on noncommutative spaces has been studied extensively, see e.g. [1] and references therein. General quantum mechanical arguments indicate that it is not possible to measure a classical background space-time at the Planck scale, due to the effects of the gravitational backreaction [2]. This has led to the belief that the classical differentiable manifold structure of space-time at the Planck scale should be replaced by some sort of noncommutative structure. The simplest approximation is a flat noncommutative space-time, which can be realized by the coordinate operators  $\hat{x}^\mu$  satisfying  $[\hat{x}^\mu, \hat{x}^\nu] = i\hbar\theta^{\mu\nu}$ , where  $\theta^{\mu\nu}$  is the noncommutativity parameter. However, the restriction to flat space-time is not natural and one must discuss more general curved noncommutative space-time, when the commutator of coordinates depends on these coordinates.

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The generalized noncommutative spaces arise e.g. in the context of string theory because of the presence of background antisymmetric magnetic  $B$ -field.

The construction of a consistent quantum field theory and gravity on a curved noncommutative space is one of the main open challenges in modern theoretical physics. However, to do it is not so easy because of the conceptual and technical problems. To begin with let us study quantum mechanics QM with position-dependent noncommutativity.

Usually, noncommutative QM [3] deals with the following commutation relations:

$$[\hat{x}^i, \hat{x}^j] = i\hbar\theta^{ij}, \quad (1)$$

$$[\hat{x}^i, \hat{p}_j] = i\hbar\delta_j^i, \quad (2)$$

$$[\hat{p}_i, \hat{p}_j] = 0, \quad (3)$$

where  $\theta^{ij}$  is some constant antisymmetric matrix. However, it is not always reasonable to assume that the noncommutativity extends to the whole space, leaving the parameter of noncommutativity  $\theta^{ij}$  to be constant. One can consider more general situation of position-dependent or even local noncommutativity, when noncommutativity exists only in some restricted area of the space, like, e.g., in the two-dimensional case,

$$[\hat{x}, \hat{y}] = \frac{i\hbar\theta}{1 + \theta\alpha(\hat{x}^2 + \hat{y}^2)}. \quad (4)$$

The constant  $\alpha$  is a parameter which measure the degree of locality, if  $\alpha = 0$  the noncommutativity is global (1-3), if  $\alpha \neq 0$  the noncommutativity is local. Other examples of position-dependent noncommutativity are Lie-algebraic  $[\hat{x}^i, \hat{x}^j] = i\hbar f_k^{ij} \hat{x}^k$  and, in particular the kappa-Poincare noncommutativity [4], and the quadratic noncommutative algebra  $[\hat{x}^i, \hat{x}^j] = i\hbar R_{kl}^{ij} \hat{x}^k \hat{x}^l$  which appears in the context of quantum groups [5], [6].

The aim of this work is to construct consistent quantum mechanics with a given position-dependent noncommutativity,

$$[\hat{x}^i, \hat{x}^j] = i\hbar\omega^{ij}(\hat{x}), \quad (5)$$

i.e., to construct the complete algebra of commutation relations, including momenta, which obey the Jacobi identity.

## 2 Jacobi identity and position-dependent noncommutativity

Note that in the presence of the position-dependent noncommutativity (5), the other commutators  $[\hat{x}^i, \hat{p}_j]$  and  $[\hat{p}_i, \hat{p}_j]$  should be changed as well in order to satisfy

the Jacobi identity. For example, consider the identity

$$[\hat{p}_k, [\hat{x}^i, \hat{x}^j]] + [\hat{x}^j, [\hat{p}_k, \hat{x}^i]] + [\hat{x}^i, [\hat{x}^j, \hat{p}_k]] \equiv 0 , \quad (6)$$

where coordinates obey (5) and momenta still obey (2), (3). Then from (6) one has:

$$[\hat{p}_k, \omega^{ij}(\hat{x})] + [\hat{x}^j, \delta_k^i] + [\hat{x}^i, \delta_k^j] \equiv 0 ,$$

or

$$[\hat{p}_k, \omega^{ij}(\hat{x})] \equiv 0 . \quad (7)$$

If we suppose now that

$$\omega^{ij}(\hat{x}) = f_l^{ij} \hat{x}^l , \quad (8)$$

then from (2) and (7) it follows that

$$[\hat{p}_k, f_l^{ij} \hat{x}^l] = -i\hbar f_l^{ij} \delta_k^l = -i\hbar f_k^{ij} \equiv 0 . \quad (9)$$

Thus, because of the Jacobi identity, the NCQM commutation relations (1-3) are valid only for a position independent parameter  $\theta^{ij}$ . Otherwise, we should change (2) and (3) as well in order to satisfy the Jacobi identity including coordinates and momenta. And the question is how to do it?

### 3 The model of position-dependent noncommutativity

To answer the question posed at the end of the previous section, let us consider the classical model described by the first-order Lagrangian

$$L = p_i \dot{x}^i - H(p, x) + (p_i + B_i(x, \alpha)) \theta^{ij} (\dot{p}_j + \dot{B}_j(x, \alpha)) / 2 , \quad (10)$$

where the functions  $B_i$  depend on the parameter  $\alpha$ , such that  $B_i \rightarrow 0$  if  $\alpha \rightarrow 0$ , and  $H(p, x)$  is a given function which we will call Hamiltonian. This Lagrangian is, in fact, a generalization of a first-order model [7] which reproduce after quantization the NCQM commutation relations (1)-(3). Note that first-order Lagrangians also have been used in the context of chiral bosons [8]. For simplicity we consider just a two dimensional case,  $i = 1, 2$ ,  $x^i = (x, y)$ ,  $p_i = (p_x, p_y)$ ,  $B_i = (B_x, B_y)$  and

$$\theta^{ij} = \theta \varepsilon^{ij} , \quad (11)$$

where  $\theta$  is a real number which, as we will see, controls the noncommutativity, and  $\varepsilon^{12} = 1$ . In the limit of  $\theta \rightarrow 0$  the action (10) transforms into the usual Hamiltonian action of classical mechanics.

The Hamiltonization and canonical quantization of theories with first-order Lagrangians were considered in [9], see also [10]. Following the general lines of [9], we construct the Hamiltonian formulation of (10). Let us first rewrite (10) as

$$L = p_i \dot{x}^i + \frac{\theta}{2} p_i \varepsilon^{ij} \dot{p}_j + \theta B_i \varepsilon^{ij} \dot{p}_j + \frac{\theta}{2} B_j \varepsilon^{jk} \partial_i B_k \dot{x}^i - H(p, x) . \quad (12)$$

We adopt the notation of [9],  $\xi^\mu = (x, y, p_x, p_y)$ ,  $J_\mu = (J_i, J_{i+2})$ , where

$$J_i = p_i + \frac{\theta}{2} B_j \varepsilon^{jk} \partial_i B_k, \quad J_{i+2} = -\frac{\theta}{2} \varepsilon^{ij} (p_j + 2B_j) .$$

In this notation (12) has the form

$$L = J_\mu \dot{\xi}^\mu - H(\xi) . \quad (13)$$

The Hamiltonization of the first-order Lagrangian (13) leads to the Hamiltonian theory with second-class constraints

$$\Phi_\mu(\xi, \pi) = \pi_\mu - J_\mu(\xi) = 0 , \quad (14)$$

where  $\pi_\mu$  are the momenta conjugated to  $\xi_\mu$ . The constraint bracket is

$$\{\Phi_\mu, \Phi_\nu\} = \Omega_{\mu\nu} = \partial_\mu J_\nu - \partial_\nu J_\mu .$$

For the canonical variables  $\xi_\mu$  the Dirac brackets are

$$\{\xi^\mu, \xi^\nu\}_D = \omega_0^{\mu\nu}, \quad \omega_0^{\mu\nu} = \Omega_{\mu\nu}^{-1} .$$

The explicit form is:

$$\begin{aligned} \{x^i, x^j\}_D &= \theta d \varepsilon^{ij}, \\ \{x^i, p_j\}_D &= d (\delta_j^i - \theta \varepsilon^{ik} \partial_k B_j), \\ \{p_i, p_j\}_D &= \theta (\partial_2 B_2 \partial_1 B_1 - \partial_1 B_2 \partial_2 B_1) d \varepsilon_{ij}, \end{aligned} \quad (15)$$

where

$$d = \frac{1}{1 + \theta (\partial_1 B_2 - \partial_2 B_1)} . \quad (16)$$

It is easy to see that in the commutative limit,  $\theta \rightarrow 0$ , the constructed Dirac brackets (15) transform into the canonical Poisson brackets  $\{x^i, x^j\} = \{p_i, p_j\} = 0$ ,  $\{x^i, p_j\} = \delta_j^i$ , and in the limit  $\alpha \rightarrow 0$  ( $B_i \rightarrow 0$ ), (15) transform into

$$\{x^i, x^j\}_D = \theta \varepsilon^{ij}, \quad \{x^i, p_j\}_D = \delta_j^i, \quad \{p_i, p_j\}_D = 0,$$

which will reproduce after quantization NCQM commutation relations (1)-(3). So, in the general case, the vector field  $B_i$  introduced in order to generalize the previously known model [7], can be interpreted as the correction to the symplectic

potential which measure the curvature of the phase space due to noncommutativity.

At this point we may ask if it is possible to generalize the above construction to the case of second order models, i.e., models whose Lagrangians are quadratic in the velocities. To investigate this possibility we consider the model introduced by Lukierski et al [11]:

$$L_{LSZ} = \frac{\dot{x}_i^2}{2} + \frac{\theta}{2}\varepsilon_{ij}\dot{x}_i\ddot{x}_j. \quad (17)$$

Introducing Lagrangian multipliers  $p_i$  and new variables  $y_i$ , one rewrites (17) in an equivalent form:

$$L^{(0)} = p_i(\dot{x}_i - \dot{y}_i) + \frac{y_i^2}{2} + \frac{\theta}{2}\varepsilon_{ij}y_i\dot{y}_j. \quad (18)$$

Next, by using the Horvathy-Plyushchay variables [12]

$$X_i = x_i + \theta\varepsilon_{ij}y_j - \theta\varepsilon_{ij}p_j, \quad Q_i = \theta(y_i - p_i), \quad (19)$$

we represent (18) as

$$L^{(0)} = L_{ext}^{(0)} + L_{int}^{(0)}, \quad (20)$$

where

$$\begin{aligned} L_{ext}^{(0)} &= p_i\dot{X}_i + \frac{\theta}{2}\varepsilon_{ij}p_i\dot{p}_j - \frac{1}{2}p_i^2, \\ L_{int}^{(0)} &= \frac{1}{2\theta}\varepsilon_{ij}Q_i\dot{Q}_j + \frac{1}{2\theta^2}Q_i^2. \end{aligned}$$

We see that Lagrangian (20) separates into two disconnected parts describing the “external” and “internal” degrees of freedom. The Lagrangian  $L_{ext}^{(0)}$  is exactly a first-order model [7] for which we construct the generalization (10). Note that if now to put in (20) instead  $L_{ext}^{(0)}$  the generalized Lagrangian (10) and then to make an inverse transformation to (19) (to turn back from the Horvathy-Plyushchay variables to the original ones) we will come to a Lagrangian involving time derivatives of variables  $p_i$ . So,  $p_i$  are not Lagrangian multipliers any more and cannot be eliminated from consideration in order to go back to the higher order model (17). Therefore, the generalization to the case of an arbitrary fields  $B_i$  is possible only in the first-order model [7].

## 4 Quantization

After canonical quantization, the Dirac brackets (15) will determine the commutation relations between the operators of the coordinates and momenta  $\hat{\xi}^\mu = (\hat{x}, \hat{y}, \hat{p}_x, \hat{p}_y)$ :

$$[\hat{\xi}^\mu, \hat{\xi}^\nu] = i\hbar\omega^{\mu\nu}(\hat{x}, \hat{y}), \quad (21)$$

and quantum Hamiltonian  $\hat{H}$  is constructed according to the classical function  $H(p, x)$ , where some ordering must be chosen in order to construct the operators  $\omega^{\mu\nu}(\hat{x}, \hat{y})$  and  $\hat{H}$ . The most natural choice is the symmetric Weyl ordering prescription, where to each function  $f(\xi)$  on the phase space is associated a symmetrically ordered operator function  $\hat{f}(\hat{\xi})$  according to the rule

$$\hat{f}(\hat{\xi}) = \int \frac{d^4 k}{(2\pi\hbar)^4} \tilde{f}(k) e^{-\frac{i}{\hbar} k_\mu \hat{x}^\mu}, \quad (22)$$

with  $\tilde{f}(k)$  is a Fourier transform of  $f$ . In particular, the function  $d(x, y)$  will determine the position-dependent noncommutativity,  $[\hat{x}, \hat{y}] = i\hbar\theta d(\hat{x}, \hat{y})$ .

In [13] it was shown that the Jacobi identity for the operator algebra (21) is equivalent to the following condition

$$(\xi^\mu \star \omega^{\nu\lambda} - \omega^{\nu\lambda} \star \xi^\mu) + \text{cycl}(\mu\nu\lambda) = 0, \quad (23)$$

where

$$f \star g = \sum_{k=0}^{\infty} \hbar^k f \star_k g = f \cdot g + \frac{i\hbar}{2} \omega^{\mu\nu} \partial_\mu f \partial_\nu g + \dots \quad (24)$$

is a star product associated with the noncommutative algebra (21) and  $\omega^{\nu\lambda} = \omega_0^{\nu\lambda} + (\text{quantum corrections})$ . In the first order in  $\hbar$  the equation (23) is equivalent to the Jacobi identity for the classical matrix  $\omega_0^{\mu\nu}$ :

$$\omega_0^{\mu\sigma} \partial_\sigma \omega_0^{\nu\lambda} + \text{cycl}(\mu\nu\lambda) = 0, \quad (25)$$

which we have by the construction. In the second order, as well as in all even orders, the left-hand side of (23) is identically equal to zero, since

$$f \star_{2n} g - g \star_{2n} f = 0. \quad (26)$$

In the third order the condition (23) is not satisfied for  $\omega^{\mu\nu} = \omega_0^{\mu\nu}$ , i.e. it does not follow from the Jacobi identity (25) for  $\omega_0^{\mu\nu}$ . To solve this problem one can construct a quantum correction to  $\omega_0$ , and this has to be an  $\hbar^2$  correction:

$$\omega^{\mu\nu} = \omega_0^{\mu\nu} + \hbar^2 \omega_2^{\mu\nu} + O(\hbar^4). \quad (27)$$

Doing so, the third order of the condition (23) will become

$$(\xi^\mu \star_3 \omega_0^{\nu\lambda} - \omega_0^{\nu\lambda} \star_3 \xi^\mu) + (\xi^\mu \star_1 \omega_2^{\nu\lambda} - \omega_2^{\nu\lambda} \star_1 \xi^\mu) + \text{cycl}(\mu\nu\lambda) = 0. \quad (28)$$

A quantum non-Poisson correction  $\omega_2^{\mu\nu}$  can be found from (28) and has the form:

$$\omega_2^{\mu\nu} = \frac{1}{48} \partial_\gamma \omega_0^{\rho\sigma} \partial_\rho \omega_0^{\gamma\delta} \partial_\sigma \partial_\delta \omega_0^{\mu\nu} - \frac{1}{24} \partial_\sigma \partial_\gamma \omega_0^{\mu\rho} \partial_\rho \partial_\delta \omega_0^{\nu\sigma} \omega_0^{\gamma\delta}. \quad (29)$$

An explicit formulae for  $\omega_2^{\mu\nu}$  taking into account the concrete form (15) of  $\omega_0^{\mu\nu}$  is presented in appendix. A systematic procedure for the construction of quantum corrections  $\omega_{2n}^{\mu\nu}$  to the classical Dirac bracket  $\omega_0^{\mu\nu}$  was described in [13], but explicit calculations were made only up to the fourth order in  $\hbar$  and no general formula is yet available.

Note that in some particular cases in which there is no ordering problem, e.g., for a linear Poisson structure  $\omega^{\mu\nu}$  or if  $\omega^{\mu\nu}$  depends only on one of the coordinates, the quantum Dirac brackets  $\omega^{\mu\nu}$  coincide with the classical ones  $\omega_0^{\mu\nu}$  (there is no corrections). In this case, the Jacobi identity for the quantum algebra (21) holds true as a consequence of the Jacobi identity for the matrix  $\omega_0^{\mu\nu}(x, y)$ . The interesting question is whether it is possible to present an exact formulae for quantized Dirac brackets of the model or one can only get some reasonable approximation, expressed as power series in  $\hbar$ ?

To work with operators  $\hat{\xi}^\mu$  which obey the commutation relations (21) one can use the polydifferential representation of the algebra (21):  $\hat{\xi}^\mu = \xi^\mu + i\hbar/2\omega^{\mu\nu}\partial_\nu + \dots$ , constructed in [13].

## 5 Definition of $B_i$

Suppose that we know the position-dependent noncommutativity from some physical considerations, i.e., the function  $d(x, y)$ , which is the Weyl symbol of the operator  $d(\hat{x}, \hat{y})$ , is given. In order to define the complete algebra (15), we need to know the functions  $B_i$ . For that one can use the equation (16). However, one cannot determine two functions  $B_x$  and  $B_y$  from just one equation (16). Therefore, we need to impose one additional condition. We will consider now two different choices of the additional conditions.

Let us first consider the condition  $B_i = \varepsilon^{ij}\partial_j\phi$ , so that the equation (16) becomes

$$d = \frac{1}{1 + \theta \Delta \phi} ,$$

where  $\Delta = \partial_x^2 + \partial_y^2$ . Suppose that the function  $d$  has a rotational symmetry like in the example (4), i.e.,

$$d = \frac{1}{1 + \theta f(\alpha(x^2 + y^2))} , \quad (30)$$

where  $f$  is some given function,  $f(0) = \text{const} < \infty$ . We will also need the integral  $F$ ,  $F' = f$ ,  $F(0) = \text{const} < \infty$ .

From (16) and (30) one finds

$$\Delta \phi = f(\alpha(x^2 + y^2)) . \quad (31)$$

In polar coordinates  $x = r \cos \varphi$ ,  $y = r \sin \varphi$  the equation (31) can be written as:

$$\frac{1}{r} \partial_r r \partial_r \phi = f(\alpha r^2) , \quad (32)$$

which yields

$$\partial_r \phi = \frac{F(\alpha r^2)}{2\alpha r} + \frac{c}{r}. \quad (33)$$

We fix the constant  $c$  from the condition

$$\lim_{\alpha \rightarrow 0} \partial_r \phi = 0 \quad (34)$$

which gives  $c = -\frac{F(0)}{2\alpha}$

$$\partial_r \phi = \frac{F(\alpha r^2) - F(0)}{2\alpha r}. \quad (35)$$

Then we calculate

$$\begin{aligned} B_x = \partial_y \phi &= \left( \sin \varphi \partial_r + \frac{1}{r} \cos \varphi \partial_\varphi \right) \phi(r) = \\ \sin \varphi \frac{F(\alpha r^2) - F(0)}{2\alpha r} &= y \frac{F(\alpha(x^2 + y^2)) - F(0)}{2\alpha(x^2 + y^2)}, \end{aligned} \quad (36)$$

and

$$\begin{aligned} B_y = -\partial_x \phi &= -\left( \cos \varphi \partial_r - \frac{1}{r} \sin \varphi \partial_\varphi \right) \phi(r) = \\ -\cos \varphi \frac{F(\alpha r^2) - F(0)}{2\alpha r} &= -x \frac{F(\alpha(x^2 + y^2)) - F(0)}{2\alpha(x^2 + y^2)}. \end{aligned} \quad (37)$$

We see that  $B_i \rightarrow 0$  when  $\alpha \rightarrow 0$ .

The second choice is  $B_x = B_y = \chi$ . Note, that this condition implies that  $\{p_x, p_y\}_D = 0$ . We consider more general case

$$d = \frac{1}{1 + \theta g(\alpha, x, y)},$$

where  $g(\alpha, x, y)$  is an arbitrary function,  $g(0, x, y) = 0$ . The equation (16) yields

$$(\partial_x - \partial_y) \chi = g(\alpha, x, y)$$

After the change of variables  $\xi = x - y$ ,  $\eta = x + y$ , one has

$$\partial_\xi \chi = g\left(\alpha, \frac{1}{2}(\xi + \eta), \frac{1}{2}(\xi - \eta)\right),$$

the solution of this equation is

$$\chi = G_\xi(\xi, \eta) + G_0(\eta)$$

where

$$G_\xi(\xi, \eta) = \int d\xi g\left(\alpha, \frac{1}{2}(\xi + \eta), \frac{1}{2}(\xi - \eta)\right),$$



and the function  $G_0(\eta)$  can be determined from the condition that  $\lim_{\alpha \rightarrow 0} \chi = 0$ .

Thus, we have constructed the classical model (10) which after quantization leads to the two-dimensional QM with position-dependent noncommutativity  $[\hat{x}, \hat{y}] = i\theta d(\hat{x}, \hat{y})$ . To define this model we use the position-dependent noncommutativity itself, which is supposed to be known ab initio, and an additional condition, imposed by hand from some physical considerations. For example, if we want  $[\hat{p}_x, \hat{p}_y] = 0$ , we choose the additional condition  $B_x = B_y$ , etc.

## 6 Local noncommutativity

Let us consider the particular example of local noncommutativity (4). In this case the function  $d$  is

$$d = \frac{1}{1 + \theta\alpha(x^2 + y^2)}.$$

The first choice of additional condition ( $B_i = \varepsilon^{ij}\partial_j\phi$ ) implies:

$$B_x = -\frac{\alpha}{4}y(x^2 + y^2), \quad B_y = \frac{\alpha}{4}x(x^2 + y^2),$$

and the Dirac brackets (15) are

$$\begin{aligned} \{x, y\}_D &= \theta d, \quad \{p_x, p_y\}_D = \frac{3\theta\alpha^2}{16}(x^2 + y^2)^2 d, \\ \{x, p_x\}_D &= \left[1 + \frac{\alpha\theta}{4}(x^2 + 3y^2)\right] d, \quad \{x, p_y\}_D = -\frac{\alpha\theta}{2}xyd, \\ \{y, p_y\}_D &= \left[1 + \frac{\alpha\theta}{4}(3x^2 + y^2)\right] d, \quad \{y, p_x\}_D = -\frac{\alpha\theta}{2}xyd. \end{aligned} \tag{38}$$

The second choice means

$$B_x = B_y = \frac{\alpha}{3}(x^3 - y^3),$$

and

$$\begin{aligned} \{x, y\}_D &= \theta d, \quad \{p_x, p_y\}_D = 0, \\ \{x, p_x\}_D &= [1 + \alpha\theta y^2] d, \quad \{x, p_y\}_D = \alpha\theta x^2 d, \\ \{y, p_y\}_D &= [1 + \alpha\theta x^2] d, \quad \{y, p_x\}_D = \alpha\theta y^2 d. \end{aligned} \tag{39}$$

In order to compare the two models we consider the limit  $r \rightarrow \infty$ . In both cases  $\{x, y\}_D \rightarrow 0$  and the Dirac brackets  $\{x, p_x\}_D$ ,  $\{x, p_y\}_D$ ,  $\{y, p_y\}_D$  and  $\{y, p_x\}_D$  are limited functions in this limit. However,  $\lim_{r \rightarrow \infty} \{p_x, p_y\}_D = \infty$  in the first model, while  $\{p_x, p_y\}_D = 0$  in the second. Since, usually, the non-zero commutator of the momenta means the presence of a magnetic field, it would be difficult to give some physical meaning to the first model on the infinity whereas the second one is free from this difficulty.

## 7 Discussions and conclusions

We have proposed a model of the consistent quantum mechanics with position-dependent noncommutativity. Our construction is based on the first-order Lagrangian, which after quantization reproduces the desired commutation relations between the operators of coordinates and momenta.

Note that a first-order Lagrangian for the Duval-Horvathy model [14] can also lead to the position-dependent Dirac brackets [15], see also [16], where the correspondent symplectic structure was obtained by means of introducing an interaction with the magnetic field in the model of nonrelativistic anyon [17]. However, the position-dependence in this case is due to the presence of a nonconstant magnetic field  $B(x)$ . In our model (10) the noncommutativity is caused by other factors and magnetic field can enter the theory via Hamiltonian  $H(x, p)$ . Also, the possibility to localize the noncommutativity within the model [4] meets some difficulties, since the magnetic field  $B(x)$  should go to infinity outside the area of local noncommutativity. Three-dimensional generalization of the model [4] was considered in [18].

It should be mentioned that the particular case of a position-dependent noncommutativity, a model of a point particle on kappa-Minkowski space was derived from a first-order Lagrangian in [19].

In order to obtain some phenomenological consequences of such a type of noncommutativity in space it would be interesting to consider some particular physical problems in the presence of this noncommutativity. For example, the scattering of plane waves on the local noncommutativity. For that one needs to take the Hamiltonian of free particle  $\hat{H} = \frac{1}{2}(\hat{p}_x^2 + \hat{p}_y^2)$  and to use perturbation theory on  $\theta$ . Also, it would be interesting to calculate the uncertainty relations.

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## 8 Appendix

Taking into account the concrete form (15) of  $\omega_0^{\mu\nu}$  one can calculate the explicit form of quantum non-Poisson correction  $\omega_2^{\mu\nu}$ , which are listed below with  $\mu < \nu$ :

$$\begin{aligned}
\omega_2^{12} &= \frac{\theta^3}{24} \left[ \frac{1}{2} (\partial_2 d)^2 \partial_1^2 d - \partial_1 d \partial_2 d \partial_1 \partial_2 d + \frac{1}{2} (\partial_1 d)^2 \partial_2^2 d \right. \\
&\quad \left. + d (\partial_1 \partial_2 d)^2 - d \partial_2^2 d \partial_1^2 d \right], \\
\omega_2^{ij+2} &= \frac{\theta^2}{24} \left[ \frac{1}{2} (\partial_2 d)^2 \partial_1^2 - \partial_1 d \partial_2 d \partial_1 \partial_2 + \frac{1}{2} (\partial_1 d)^2 \partial_2^2 \right] \\
&\quad \times (\delta_j^i d - \theta \varepsilon^{ik} \partial_k B_j d) - \frac{\theta^2}{24} \varepsilon^{im} d (\partial_j \partial_1 d \partial_m \partial_2 d - \partial_j \partial_2 d \partial_m \partial_1 d) \\
&\quad + \frac{\theta^3}{24} \varepsilon^{im} \varepsilon^{jk} d [\partial_n \partial_1 d \partial_m \partial_2 (\partial_k B_n d) - \partial_n \partial_2 d \partial_m \partial_1 (\partial_k B_n d)], \\
\omega_2^{34} &= \frac{\theta^3}{24} \left[ \frac{1}{2} (\partial_2 d)^2 \partial_1^2 - \partial_1 d \partial_2 d \partial_1 \partial_2 + \frac{1}{2} (\partial_1 d)^2 \partial_2^2 \right] \\
&\quad \times ((\partial_2 B_2 \partial_1 B_1 - \partial_1 B_2 \partial_2 B_1) d) - \\
&\quad \frac{\theta}{24} d [\partial_n \partial_1 (\delta_m^1 d - \theta \partial_2 B_m d) \partial_m \partial_2 (\delta_n^2 d + \theta \partial_1 B_n d) \\
&\quad - \partial_n \partial_2 (\delta_m^1 d - \theta \partial_2 B_m d) \partial_m \partial_1 (\delta_n^2 d - \theta \partial_1 B_n d)].
\end{aligned}$$

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